



6.4.1 Matrices

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Introduction

- In this topic, we will
 - Define matrices and its dimensions
 - Describe the rows and columns of a matrix and the column vectors
 - Explain how to identify specific entries
 - Define the diagonal, super-diagonal and sub-diagonal entries





Matrix representation

- We have now seen that these two problems are equivalent:
 - Given this system of linear equations

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$

we can write it as

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \quad \mathbf{a}_j = \begin{pmatrix} a_{1,j} \\ a_{2,j} \\ a_{3,j} \\ \vdots \\ a_{m,j} \end{pmatrix}$$





Matrix representation

- We will now define a matrix A that amalgamates the coefficients, so given

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$

define the *matrix* $A =$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}$$





Matrix representation

- For example,

$$3x_1 + 4x_2 - x_3 = 5$$

$$5x_1 + 7x_2 = 9$$

$$2x_1 - 6x_2 + 8x_3 = 4$$

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 5 & 7 & 0 \\ 2 & -6 & 8 \end{pmatrix}$$

$$-x_1 - 2x_2 + 3x_3 + 8x_4 = 0$$

$$7x_1 - 4x_2 + 5x_3 - 6x_4 = 9$$

$$B = \begin{pmatrix} -1 & -2 & 3 & 8 \\ 7 & -4 & 5 & -6 \end{pmatrix}$$

$$1.3x_1 - 2.8x_2 = 1$$

$$-6.4x_1 - 0.9x_2 = 2$$

$$5.7x_1 + 3.2x_2 = 3$$

$$8.0x_1 - 4.5x_2 = 4$$

$$7.1x_1 + 9.6x_2 = 5$$

$$C = \begin{pmatrix} 1.3 & -2.8 \\ -6.4 & -0.9 \\ 5.7 & 3.2 \\ 8.0 & -4.5 \\ 7.1 & 9.6 \end{pmatrix}$$





Dimensions

- The dimensions of a matrix corresponding to a system of m linear equations in n unknowns is said to be $m \times n$

$$\begin{array}{c} \updownarrow \\ m \end{array} A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix} \begin{array}{c} \leftarrow \cdots \rightarrow \\ n \end{array}$$

- A matrix is said to have m rows
and n columns





Indexing

- Individual entries of a matrix may be identified by a pair of indices
 $a_{i,j}$ represents the entry in Row i and Column j in the matrix A

Row 2 $\longrightarrow$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix}$$

Column 3 $\uparrow$





Indexing

- Here is a 7×6 matrix

$$\begin{array}{l} c_{3,2} = 8.2 \\ c_{1,4} = -9.1 \\ c_{3,2} = -8.3 \\ c_{6,6} = 9.2 \\ c_{4,4} = 7.6 \end{array} \quad C = \begin{pmatrix} 3.2 & -7.5 & 1.2 & -4.5 & -9.1 & 0.5 \\ 6.8 & -4.5 & 2.9 & 3.9 & 2.4 & 0.7 \\ -7.4 & 8.2 & 6.0 & 2.1 & 5.4 & 1.9 \\ 1.0 & 5.1 & 8.2 & 7.6 & 6.3 & 4.9 \\ 9.5 & 0.3 & 3.7 & 5.0 & -7.8 & 2.6 \\ 8.7 & 5.4 & 9.1 & -8.4 & 0.5 & 9.2 \\ -8.3 & 3.8 & 7.3 & 1.2 & 4.6 & 0.0 \end{pmatrix}$$





Indexing

- My aide memoire: The location of $c_{i,j}$

$$C = \begin{pmatrix} 3.2 & -7.5 & 1.2 & -4.5 & -9.1 & 0.5 \\ 6.8 & -4.5 & 2.9 & 3.9 & 2.4 & 0.7 \\ -7.4 & 8.2 & 6.0 & 2.1 & 5.4 & 1.9 \\ 1.0 & 5.1 & 8.2 & 7.6 & 6.3 & 4.9 \\ 9.5 & 0.3 & 3.7 & 5.0 & -7.8 & 2.6 \\ 8.7 & 5.4 & 9.1 & -8.4 & 0.5 & 9.2 \\ -8.3 & 3.8 & 7.3 & 1.2 & 4.6 & 0.0 \end{pmatrix}$$

“Down the stairs, and into the crypt.”





Column vectors of a matrix

- Because a system of linear equations is equivalent to finding a linear combination of vectors equaling a target vector, we will identify the columns as vectors

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n} \end{pmatrix} \quad \mathbf{a}_1 = \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \vdots \\ a_{m,1} \end{pmatrix}, \quad \dots \quad \mathbf{a}_n = \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \\ \vdots \\ a_{m,n} \end{pmatrix}$$
$$= (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \cdots \ \mathbf{a}_n)$$

- The columns of matrices A , B and C are $\mathbf{a}_j$, $\mathbf{b}_j$ and $\mathbf{c}_j$, respectively
- The entries of these matrices are $a_{i,j}$, $b_{i,j}$ and $c_{i,j}$, respectively





Column vectors of a matrix

- For example, given the matrix

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 5 & 7 & 0 \\ 2 & -6 & 8 \end{pmatrix}$$

the column vectors are

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} \quad \mathbf{a}_2 = \begin{pmatrix} 4 \\ 7 \\ -6 \end{pmatrix} \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \\ 8 \end{pmatrix}$$





Square matrices

- When $m = n$, for obvious reasons, we call a matrix a *square matrix*

$$A = \begin{pmatrix} 5.7 & 1.3 \\ 1.8 & 6.4 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 4 & -1 \\ 5 & 7 & 0 \\ 2 & -6 & 8 \end{pmatrix}$$

$$C = \begin{pmatrix} 8.2 & -0.7 & 1.2 & 0.5 & 0.1 \\ 1.3 & 7.4 & 1.7 & 0.9 & 0.2 \\ 0.5 & 1.8 & 6.8 & 1.2 & 0.6 \\ -0.3 & 0.7 & 1.5 & 9.3 & 1.4 \\ -0.2 & 0.1 & 0.8 & 1.1 & 10.3 \end{pmatrix}$$





Diagonal entries

- The diagonal entries of a matrix are those which indices are equal

$$c_{1,1} = 3.2$$

$$c_{2,2} = -4.5$$

$$c_{3,3} = 6.0$$

$$c_{4,4} = 7.6$$

$$c_{5,5} = -7.8$$

$$c_{6,6} = 9.2$$

$$C = \begin{pmatrix} 3.2 & -7.5 & 1.2 & -4.5 & -9.1 & 0.5 \\ 6.8 & -4.5 & 2.9 & 3.9 & 2.4 & 0.7 \\ -7.4 & 8.2 & 6.0 & 2.1 & 5.4 & 1.9 \\ 1.0 & 5.1 & 8.2 & 7.6 & 6.3 & 4.9 \\ 9.5 & 0.3 & 3.7 & 5.0 & -7.8 & 2.6 \\ 8.7 & 5.4 & 9.1 & -8.4 & 0.5 & 9.2 \\ -8.3 & 3.8 & 7.3 & 1.2 & 4.6 & 0.0 \end{pmatrix}$$

- We often will refer to diagonal entries $c_{i,i}$





Off-diagonal entries

- All entries not on the diagonal are called *off-diagonal entries*

$$C = \begin{pmatrix} 3.2 & -7.5 & 1.2 & -4.5 & -9.1 & 0.5 \\ 6.8 & -4.5 & 2.9 & 3.9 & 2.4 & 0.7 \\ -7.4 & 8.2 & 6.0 & 2.1 & 5.4 & 1.9 \\ 1.0 & 5.1 & 8.2 & 7.6 & 6.3 & 4.9 \\ 9.5 & 0.3 & 3.7 & 5.0 & -7.8 & 2.6 \\ 8.7 & 5.4 & 9.1 & -8.4 & 0.5 & 9.2 \\ -8.3 & 3.8 & 7.3 & 1.2 & 4.6 & 0.0 \end{pmatrix}$$





Super-diagonal entries

- All entries immediately to the right and above the diagonal are said to be on the *super-diagonal*

$$C = \begin{pmatrix} 3.2 & -7.5 & 1.2 & -4.5 & -9.1 & 0.5 \\ 6.8 & -4.5 & 2.9 & 3.9 & 2.4 & 0.7 \\ -7.4 & 8.2 & 6.0 & 2.1 & 5.4 & 1.9 \\ 1.0 & 5.1 & 8.2 & 7.6 & 6.3 & 4.9 \\ 9.5 & 0.3 & 3.7 & 5.0 & -7.8 & 2.6 \\ 8.7 & 5.4 & 9.1 & -8.4 & 0.5 & 9.2 \\ -8.3 & 3.8 & 7.3 & 1.2 & 4.6 & 0.0 \end{pmatrix}$$





Sub-diagonal entries

- All entries immediately to the left or below the diagonal are said to be on the *sub-diagonal*

$$C = \begin{pmatrix} 3.2 & -7.5 & 1.2 & -4.5 & -9.1 & 0.5 \\ 6.8 & -4.5 & 2.9 & 3.9 & 2.4 & 0.7 \\ -7.4 & 8.2 & 6.0 & 2.1 & 5.4 & 1.9 \\ 1.0 & 5.1 & 8.2 & 7.6 & 6.3 & 4.9 \\ 9.5 & 0.3 & 3.7 & 5.0 & -7.8 & 2.6 \\ 8.7 & 5.4 & 9.1 & -8.4 & 0.5 & 9.2 \\ -8.3 & 3.8 & 7.3 & 1.2 & 4.6 & 0.0 \end{pmatrix}$$





Square matrices

- In a square matrix, it is easier to identify the
 - Diagonal entries
 - Super-diagonal entries
 - Sub-diagonal entries

$$A = \begin{pmatrix} 8.2 & -0.7 & 1.2 & 0.5 & 0.1 \\ 1.3 & 7.4 & 1.7 & 0.9 & 0.2 \\ 0.5 & 1.8 & 6.8 & 1.2 & 0.6 \\ -0.3 & 0.7 & 1.5 & 9.3 & 1.4 \\ -0.2 & 0.1 & 0.8 & 1.1 & 10.3 \end{pmatrix}$$





Zero matrix

- Like the zero vector, the $m \times n$ matrix of all zeros is described as the $m \times n$ zero matrix
 - It is denoted with an italicized O subscripted with m, n

$$O_{m,n} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$





Summary

- Following this topic, you now
 - Understand the definition of a matrix
 - Know the dimensions are the number of rows by the number of columns
 - If these are equal, it is a square matrix
 - Know where the entry $a_{5,7}$ is in a matrix
 - Know which are the diagonal, super-diagonal and sub-diagonal entries of a matrix
 - Understand the ideal of the zero matrix $O_{m,n}$





References

- [1] [https://en.wikipedia.org/wiki/Matrix_\(mathematics\)](https://en.wikipedia.org/wiki/Matrix_(mathematics))





Acknowledgments

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Colophon

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